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Internal Symmetry Cannot Be Concealed*

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We consider the problem of determining whether a given "particle" is elementary or is some member of a degenerate multiplet corresponding to an exact symmetry. We discuss a double-scattering experiment which can make this determination *unless* the members of the multiplet are unable to distinguish one another. Finally, such an inability is shown to be incompatible with the usual relation between spin and statistics and known analyticity properties of scattering amplitudes for strongly interacting particles.

1. INTRODUCTION

In discussions of internal symmetry the question arises, "How could we know if something which we regard as an elementary particle actually has an additional internal quantum number?" Of course if the symmetry is broken, as the proton charge breaks the isotopic spin symmetry of the proton–neutron doublet, there is no problem. Suppose, however, that the internal symmetry is *not* broken by any interactions, so that the generators of it commute with the time development operator H. How can we tell that a "hidden degeneracy" exists? Two partial answers to this question seem to be widely known. First, if the particles in question are fermions one can *imagine* confining them to a box and measuring the Fermi energy as a function of particle number. The hidden degeneracy will of course lower the Fermi energy. Second, there is always the possibility that the members of the multiplet can distinguish *each other* and the degeneracy is then revealed by a suitable double-scattering experiment. We show that the analyticity of scattering amplitudes as a function of center-of-mass scattering angle requires that this possibility is the *only* one, provided that the particles have definite statistics.

The remainder of this paper is divided into three sections. In Sec. 2 we analyze the relevant double-scattering experiment. In Sec. 3 we apply the constraints imposed by analyticity and definite statistics. A brief discussion is given in Sec. 4.

2. HYPOTHETICAL DOUBLE-SCATTERING EXPERIMENT

If something which we regard as an elementary particle has a hidden internal quantum number then every such particle in a beam or target must be described by a density matrix[1] as far as the internal quantum number is concerned. Similarly a 2-particle system will be described by a density matrix. It is convenient to represent this matrix in terms of the representations of the internal symmetry group. We assume that the multiplet to be discovered transforms according to some representation D of the internal symmetry group. We expand the density matrix in terms of the projection operators for states transforming according to all the representations $D^{(I)}$ which appear in the Clebsch–Gordan series which reduces $D \otimes D$. Let $\mathfrak{T}_{I,i}$ be the projector corresponding to the ith partner of the I representation. Thus we write

$$\rho_{\text{initial}} = \sum_{I,i} \omega(I, i)\mathfrak{T}_{I,i}. \tag{1}$$

We normalize to unit flux, setting

$$\sum_{I,i} \omega(I, i) = 1. \tag{2}$$

Now let $T(I, i, E, z)$ be the scattering amplitude in the state (I, i) corresponding to center-of-mass total energy E and scattering angle $\theta, z = \cos\theta$. After a single scattering the density matrix is

$$\rho_{\text{once}} = \sum_{I,i} \omega(I, i)\mathfrak{T}_{I,i}\,|T(I, i, E, z)|^2$$

and the corresponding flux is

$$\mathcal{F}_{\text{once}} = \sum_{I,i} \omega(I, i)\,|T(1, i, E, z)|^2. \tag{3}$$

We now imagine that the scattered and recoiling particles are transported in such a way that they collide again. If no particles are lost from the beam this will not change the density matrix (because the symmetry is exact and all "external" fields are invariant under it). Now let the particles undergo a second scattering at the *same* center-of-mass energy and scattering angle. The resulting density matrix is

$$\rho_{\text{twice}} = \sum_{I,i} \omega(I, i)\mathfrak{T}_{I,i}\,|T(I, i, E, z)|^4 \tag{4}$$

and

$$\mathcal{F}_{\text{twice}} = \sum_{I,i} \omega(I, i)\,|T(I, i, E, z)|^4. \tag{5}$$

For compactness we will set

$$\lambda(I, i) \equiv |T(I, i, E, z)|^2 \geq 0. \tag{6}$$

Now consider the case where D is 1 dimensional (that is, no hidden degeneracy). Then each sum reduces to one term and we have

$$\mathcal{F}_{\text{twice}} = (\mathcal{F}_{\text{once}})^2. \quad (7)$$

Suppose that the degenerate case manages to simulate this result. The necessary condition is

$$\sum \omega(I, i)\lambda(I, i)^2 = [\sum \omega(I, i)\lambda(I, i)]^2. \quad (8)$$

Making use of Eq. (2) we can rewrite this as

$$\sum_{\substack{I,i\\J,j}} \omega(I, i)\omega(J, j)[\lambda(I, i)^2 - \lambda(I, i)\lambda(J, j)] = 0. \quad (9)$$

Interchanging indices and adding the resulting equations we find that

$$\sum \omega(I, i)\omega(J, j)[\lambda(I, i) - \lambda(J, j)]^2 = 0. \quad (10)$$

Thus the necessary condition (8) requires that $|T(I, i)|^2$ be independent of i (which is guaranteed by the symmetry) and also independent of I. We now show that in the degenerate case this last constraint is in fact impossible.

3. STATISTICS AND ANALYTICITY

To show that the functions $T(I, E, z)$ cannot all have the same magnitude we need two facts. First, at each value of E, every $T(I, E, z)$ is an analytic function[2] of z in some neighborhood of the point $z = 0$. Hence the ratio

$$\phi_{IJ}(z) \equiv T(I, E, z)/T(J, E, z) \quad (11)$$

is analytic in some neighborhood, except possibly for poles. The condition of indistinguishability requires that ϕ have unit modulus on the real axis between $z = -1$ and $z = +1$. This means that ϕ cannot be an *odd* function. The fact that $|\phi| = 1$ would require $z = 0$ to be a singular point, but in the neighborhood of a pole an analytic function must grow without bound.

Now we show that, if there is more than one amplitude $T(I, E, z)$, at least one of the functions ϕ_{IJ} must be odd and hence cannot have unit modulus. To do this we note first that, using the usual relation between spin and statistics[3] we can require the amplitude to have definite transformation properties under exchange of the space, spin, *and* internal coordinates of the final state particles. For the spin singlet-to-singlet transition amplitude this requires that the amplitude be even under simultaneous interchange of space coordinates ($z \to -z$) and internal symmetry coordinates.

Since the generators of internal symmetry for two independent particles commute we can take each of the internal symmetry wavefunctions labeled by I to be either even or odd under interchange of internal symmetry coordinates. To see that both types must occur simply consider the (unreduced) tensor product $D \otimes D$ resolved into odd and even parts,

$$\mathfrak{D}^{(\pm)}_{ik;jl}(g) = D_{ik}(g)D_{jl}(g) \pm D_{jk}(g)D_{il}(g). \quad (12)$$

If either part vanishes we may multiply the resulting equation by $[D^{-1}(g)]_{mi}$ and sum on i to obtain

$$\delta_{mk}D_{jl}(g) \pm \delta_{ml}D_{jk}(g) = 0. \quad (13)$$

If the representation D is more than 1 dimensional we may choose $l \neq m = k$ and prove that $D_{jl}(g) = 0$, for all j and l.

Thus, there must be at least one even and one odd T_I and their ratio ϕ is subject to the analysis given above, showing that the T_I cannot all have the same modulus and proving that the internal degeneracy cannot be concealed.

4. DISCUSSION

The analysis presented here is, I think, of no practical significance. The only particles for which we can readily imagine performing the experiment of Sec. 2 are protons, and the success of the nuclear shell model makes it already clear that there is no degeneracy. It however, is of some philosophical interest to see that the degeneracy can never fully conceal itself and it is intriguing that such powerful tools enter into the proof.

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[1] An elementary discussion of the density matrix is given by A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961), pp. 331 *et seq.*

[2] This is a consequence of the axioms of field theory. H. Lehmann, Nuovo Cimento **10**, 579 (1958).

[3] The standard reference for all such questions is R. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964), p. 146.