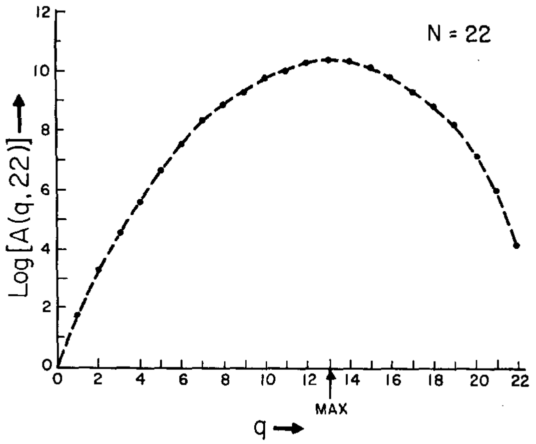

FIG. 5. A graph showing the logarithm of the number arrangements of q dumbbells on a 2×22 for all possible q. There are approximately 1.9×10^{10} ways of arranging 13 dumbbells on a 2×22 array.

For large values of N, Eq. (24) yields

$$k_0 N S_1^N = 3k_0(N-1)S_1^{N-1} + k_0(N-2)S_1^{N-2} - k_0(N-3)S_1^{N-3} + 2S_1^{N-1} + S_1^{N-2} - 3S_1^{N-3}. \quad (25)$$

We may divide this by S_1^N and, noting that

$$S_1^N = 3S_1^{N-1} + S_1^{N-2} - S_1^{N-3}, \quad (26)$$

we obtain

$$\frac{1}{S_1}(2 - 3k_0) + \frac{1}{S_1^2}(1 - 2k_0) - \frac{1}{S_1^3}(3 - 3k_0) = 0 \quad (27)$$

or

$$k_0 = 1 - \frac{S_1^2 + S_1}{3S_1^2 + 2S_1 - 3} \simeq 0.6064927. \quad (28)$$

Thus, the distribution reaches a maximum when the dumbbells occupy 61% of the compartments. Figure 5 shows $A(q, 22)$ as a function of q. In this case the maximum occurs at $q = 13$ or $\theta = 0.59$.

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Extension of the Riemann–Lebesgue Lemma*

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We show that, in the limit of large λ, integrals of the form

$$H(\lambda) \equiv \int_a^b \frac{f(x)\,dx}{u(x) + e^{i\lambda x}}$$

are essentially given by $\int_{R'} [f(x)/u(x)]\,dx$ where the region R' is the union of all those subintervals in which $|u| \geq 1$. The corrections to this expression are of two kinds: terms $O(1/\lambda)$ which depend on the details of averaging to remove logarithmic singularities in $H(\lambda)$ and terms $O[(\ln \lambda)/\lambda]$. Some examples are given. If $|u| \leq 1$, the leading term in H vanishes and $H(\lambda)$ is bounded by $(\ln \lambda)/\lambda$.

I. INTRODUCTION

It is frequently useful to be able to bound integrals of the form

$$G(\lambda) = \int_a^b f(x) e^{i\lambda x}\,dx$$

for large values of the parameter λ. Typical physical problems would be the behavior of wavepackets at large times and the behavior of the Born approximation for scattering amplitudes at large energy.[1] The mathematical tool is, of course, the Riemann–Lebesgue lemma, which ensures that

$$|G(\lambda)| \leq \text{const}\, v(f) \lambda^{-1}.$$

Here $v(f)$ denotes the total variation[2] of the function f in the interval $[a, b]$.

The study of multiple scattering from several particles, using separable potentials and the closure approximation,[3] leads to expressions of the form

$$\int_R \frac{f(x)}{u(x) + e^{i\lambda x}}\, dx = H(\lambda).$$

The purpose of the present paper is to show that $H(\lambda)$ is "essentially" given by

$$\hat{H} = \int_{R'} \frac{f(x)}{u(x)}\, dx,$$

where R' is the union of all those subintervals of R in which $|u| \geq 1$.

There are four complications which we encounter.

(i) The function $H(\lambda)$ may have recurring logarithmic singularities as λ grows without bound. We deal with this by defining an averaged function of λ:

$$H_\Delta(\lambda) = \Delta^{-1} \int_{\lambda - \frac{1}{2}\Delta}^{\lambda + \frac{1}{2}\Delta} H(\lambda')\, d\lambda'.$$

Δ must be small enough so that only one singularity appears in the integrand for any value of λ. Note that the limit of $H(\lambda)$ as $\lambda \to \infty$ is independent of Δ, provided that R' is not empty.

(ii) Again because $u(x)$ may have unit modulus, the corrections to H are not quite bounded by $1/\lambda$. The best bound we can obtain is (for $\Delta < 1$)

$$|H_\Delta(\lambda) - \hat{H}| \leq C(\ln \lambda)/\lambda\, |\ln \Delta - 1|.$$

(iii) The function f must be of bounded variation, as in the Riemann–Lebesgue lemma. Our proof requires that $|u|$ satisfy the stronger condition of piecewise monotonicity. This insures that the variation of u^n for any n can be bounded *uniformly with respect to n*.

(iv) For purely technical reasons we have had to make the following additional assumptions. We do not know whether they are necessary.

For every x_0, such that $|u(x_0)| = 1$, there is a neighborhood N of x_0 contained in $[a, b]$ in which

(a) u is differentiable,

(b) $\exists\, B_2 \ni \left| \dfrac{u'(x) - u'(x_0)}{u(x) + e^{i\lambda x}} \right| < B_2$,

(c) $\exists\, D > 0 \ni \left| \dfrac{|u(x)| - |u(x_0)|}{x - x_0} \right| > D$,

(d) $\exists\, B_1 \ni \left| \dfrac{f(x) - f(x_0)}{u(x) + e^{i\lambda x}} \right| < B_1$.

Using condition (iii), we can divide $[a, b]$ into a finite number of subintervals of three different kinds:

(A) $|u| \leq R < 1$,

(B) $|u| \geq S > 1$,

(C) $R \leq |u| \leq S$ and $|u(x)| = 1$ has exactly *one* root.

In Sec. II we find the limiting form of $H(\lambda)$ for each of these cases. In Sec. III we combine these results. Finally, in Sec. IV we discuss examples and possible extensions of this technique.

Before presenting the proof we remark that the *complex* nature of $e^{i\lambda x}$ is essential for the simple form of $\hat{H}$. For example, with the real function $\sin \lambda x$, in case (B) one may show that

$$\int \frac{1}{u(x) + \sin \lambda x}\, dx = \int \frac{1}{[u(x)^2 - 1]^{\frac{1}{2}}}\, dx + O\left(\frac{1}{\lambda}\right). \quad (1)$$

II. DISCUSSION OF THREE CASES

Case A: We have the uniformly convergent expansion

$$[u(x) + e^{i\lambda x}]^{-1} = e^{-i\lambda x} \sum_{n=0}^{\infty} [u(x) e^{-i\lambda x}]^n (-1)^n.$$

Let $v(u)$ be the variation of $|u|$. Since $|u|$ is piecewise monotonic (we denote the number of "pieces" by N),

$$v(u) \leq 2NR$$

and, because of the monotonicity,

$$v(u^n) \leq 2NR^n$$

and

$$\sup |u^n| \leq R^n.$$

Hence, the function $u^n f$ is of bounded variation, and there is a constant K such that[4]

$$v(f \cdot u^n) \leq 2NR^n \max f + R^n v(f)$$
$$\leq KR^n.$$

We now use the expansion of the denominator in the integral defining $H(\lambda)$ and apply the Riemann–Lebesgue lemma to every term in the sum:

$$\int u(x)^n f(x) e^{-\lambda x(n+1)}\, dx$$
$$\leq \frac{4}{\lambda(n+1)} [v(fu^n) + \sup |u^n f|]$$
$$\leq \frac{4}{\lambda} \frac{R^n}{n+1} [v(f) + (2N+1) \sup |f|]$$
$$\equiv \frac{4}{\lambda} \frac{R^n}{n+1} D(f, N).$$

Thus

$$f(x)[u(x) + e^{i\lambda x}]^{-1}\, dx \leq \frac{4D(f, N)}{\lambda} \frac{1}{R} |\ln (1 - R)|.$$

This completes the discussion of intervals of type A. They do not contribute to the limiting value at large λ.

Case B: Suppose $|u(x)| \geq S > 1$. Then we may write

$$f(x)[u(x) + e^{i\lambda x}]^{-1} = \frac{f(x)}{u(x)}\left(1 - \frac{1/u(x)}{[1/u(x)] + e^{-i\lambda x}}\right).$$

The first term in brackets is independent of λ. The second satisfies all the hypotheses of Case A with R replaced by $1/S$. Thus

$$\int \frac{f(x)}{u(x) + e^{i\lambda x}}\, dx = \int \frac{f(x)}{u(x)}\, dx + O\left(\frac{1}{\lambda}\right).$$

Case C: We now consider the case where $|u|$ assumes the value 1 exactly once in the interval of interest. The phase is unimportant in our argument; so we assume $u(x_1) = 1$. Now we choose an $\epsilon > 0$ such that, when

$$|x - x_1| < \epsilon,$$

x lies in the neighborhood N specified in condition (iv) of Sec. I. We have

$$\left|\frac{f(x) - f(x_1)}{u(x) + e^{i\lambda x}}\right| < B_1$$

and

$$\left|\frac{u'(x) - u'(x_1)}{u(x) + e^{i\lambda x}}\right| < B_2.$$

Since the integral exhibits logarithmic singularities at $\lambda = (\pi/x_1)(2n + 1)$, we extract them with the aid of the following two relations:

$$\frac{f(x)}{u(x) + e^{i\lambda x}} = \frac{f(x) - f(x_1)}{u(x) + e^{i\lambda x}} + f(x_1)\frac{1}{u(x) + e^{i\lambda x}},$$

$$\frac{1}{u(x) + e^{i\lambda x}} = \frac{1}{u'(x) - i\lambda}\left(\frac{d}{dx} \ln [u(x) + e^{i\lambda x}]\right.$$

$$\left. = \frac{u'(x) - u'(x_1)}{u(x) + e^{i\lambda x}} - i\lambda\frac{1 + e^{i\lambda x}}{u(x) + e^{i\lambda x}}\right).$$

Now choose $\delta \leq \min \{\epsilon, 1/\lambda\}$. Note that the definition of ϵ does not depend on λ so that, as $\lambda \to \infty$, δ will eventually be $1/\lambda$. We will show that, with suitable averaging over λ,

$$\int_{x_1-\frac{1}{2}\delta}^{x_1+\frac{1}{2}\delta} f(x)[u(x) + e^{i\lambda x}]^{-1}\, dx = O\left(\frac{1}{\lambda}\right).$$

Using the first relation above, we have

$$\int \frac{f(x)}{u(x) + e^{i\lambda x}}\, dx \leq \int \frac{|f(x) - f(x_1)|}{|u(x) + e^{i\lambda x}|}\, dx$$

$$+ |f(x_1)| \left|\int_{x_1-\frac{1}{2}\delta}^{x_1+\frac{1}{2}\delta} \frac{1}{u(x) + e^{i\lambda x}}\right| dx.$$

The first term on the right-hand side is bounded by B_1/λ. By elementary geometry we have

$$\frac{1 + e^{i\lambda x}}{u(x) + e^{i\lambda x}} \leq \frac{2}{1 + u(x)} \leq 2.$$

Thus

$$\int_{x_1-\frac{1}{2}\delta}^{x_1+\frac{1}{2}\delta} \frac{f(x)}{u(x) + e^{i\lambda x}}\, dx$$

$$\leq \frac{B_1}{\lambda} + \frac{1}{\lambda}\left(\frac{B_2}{\lambda} + 2\right) |f(x_1)|$$

$$+ \frac{1}{\lambda}|f(x_1)| \left| \int \frac{d}{dx} \ln [u(x) + e^{i\lambda x}]\, dx \right|.$$

This last integral exhibits the logarithmic singularity. Since the product $\lambda\delta$ is less than 1, the imaginary part of the integral is bounded by 1. To eliminate the divergence, we average over $\lambda - \frac{1}{2}\Delta \leq \lambda' \leq \lambda + \frac{1}{2}\Delta$:

$$\text{av} \int \frac{d}{dx} \ln [u(x) + e^{i\lambda x}]\, dx \leq C(|\ln \Delta| + 1).$$

Hence the integral in these dangerous regions is bounded by $1/\lambda$, multiplied by a factor which depends in a well-defined way on the averaging over λ.

III. PROOF OF THE THEOREM

Since u has only a finite number of oscillations, we can cover the range of integration with a finite number of intervals of type A, B, or C. Since the intervals of type C are of length $1/\lambda$, the values of R and S for the intervals of type A and B will be quite close to 1. However, for sufficiently large λ, we can be sure that $1 - R$ and $S - 1$ are at least as large as D/λ, where D is defined by condition (c). Inserting this in the bounds for case A, we find

$$H_\Delta(\lambda) = \int_{\text{type B}} \frac{f(x)}{u(x)}\, dx + K_1 \frac{\ln \lambda}{\lambda} + K_2 \frac{1 + |\ln \Delta|}{\lambda}.$$

The first correction term is due to intervals of type A and B, while the second is due to intervals of type C.

IV. EXAMPLES

Integrals of the form discussed here do not arise frequently in the mathematical literature, where expressions of the form e^{ix} usually arise from contour integration. An example arising in the physical situation discussed in Sec. I will be presented elsewhere.[3]

Examples for which the limiting form of the integral is known by other means can be found by considering Fermi–Dirac integrals.

Let

$$H(\beta) = \int_0^\infty \frac{f(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1}\, d\epsilon.$$

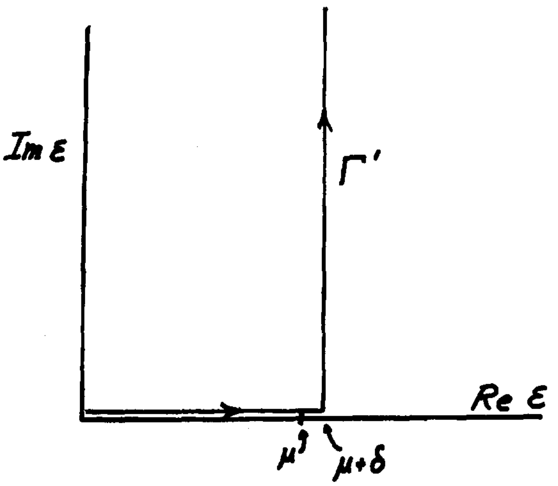

FIG. 1. The deformed contour Γ′ in the complex ε plane.

Of course, this is not yet the form in which we are interested.

However, the integrand is regular except at the singularities of f and at $\beta(\epsilon - \mu) = (2n + 1)\pi i$. Since the integrand falls as $e^{-\beta \operatorname{Re} \epsilon}$ at infinity, we can almost deform the contour of integration to the axis $\mu + iy$. We can apply our technique as follows: Let the contour Γ' be as shown in Fig. 1. Assume that f has no singularities in the first quadrant. Then

$$\int_0^\infty \frac{f(\epsilon)\, d\epsilon}{e^{(\epsilon-\mu)\beta} + 1} = \int_{\Gamma'} \frac{f(\epsilon)\, d\epsilon}{e^{(\epsilon-\mu)\beta} + 1}$$
$$= \int_0^{\mu+\delta} \frac{f(\epsilon)\, d\epsilon}{e^{(\epsilon-\mu)\beta} + 1} + i \int_0^\infty \frac{f(iy)\, dy}{e^{(\delta+iy)\beta} + 1}.$$

Assuming f to be of bounded variation on Γ', we can apply our lemma, using $u(y) = e^{-\beta\delta} < 1$. We can maintain this condition with $\delta \to 0$; for example, let $\delta = 1/\beta$. Thus

$$\int_0^\infty \frac{f(\epsilon)\, d\epsilon}{e^{(\epsilon-\mu)\beta} + 1} = \int_0^\mu \frac{f(\epsilon)\, d\epsilon}{e^{(\epsilon-\mu)\beta} + 1} + O\left(\frac{1}{\beta}\right),$$

which is the familiar result.

Further examples can be constructed by considering the integral of the exact derivative

$$\frac{d}{dx}\{g(x) \ln [u(x) + e^{i\lambda x}]\},$$

which is easily seen to be equal to

$$\frac{-i\lambda g(x)u(x)}{u(x) + e^{i\lambda x}} + i\lambda g(x) + ig'(x) \arg [u(x) + e^{i\lambda x}]$$
$$+ \frac{g(x)u'(x)}{u(x) + e^{i\lambda x}} + g'(x) \ln |u(x) + e^{i\lambda x}|.$$

If we set $g(x) = f(x)/u(x)$, the first term is seen to be an integral of the desired type multiplied by λ. Thus we examine the rest of this equation for terms of order λ. When $|u(x)| > 1$, there are none but $i\lambda g(x)$, and the integral is given by

$$\int \frac{f(x)}{u(x) + e^{i\lambda x}}\, dx \to \int g(x).$$

When $u(x) < 1$, there are several terms of order λ, with only the last two being negligible. For large λ and sufficiently smooth g, we can replace

$$\int_a^b \frac{d}{dx}\{g(x) \ln [u(x) + e^{i\lambda x}]\} \quad \text{by} \quad i\lambda g(x)x\big|_a^b$$

and

$$\arg [u(x) + e^{i\lambda x}] \quad \text{by} \quad i\lambda x.$$

A single partial integration then yields

$$i\lambda \int_a^b \frac{g(x)u(x)}{u(x) + e^{ikx}}\, dx$$
$$= i\lambda \int_a^b g(x)\, dx$$
$$+ i\lambda \int_a^b xg'(x)\, dx + O(1) - i\lambda g(x)x\big|_a^b$$
$$= 0 + O(1).$$

Of course, when $u(x)$ can assume the value 1, the logarithmic correction enters in several terms.

This "example" serves as an analog to the usual heuristic proof of the Riemann–Lebesgue lemma based on partial integration. It would be nice to find some g and u for which every term can be integrated in terms of elementary functions, but I have been unable to do so.

V. DISCUSSION

By extending the Riemann–Lebesgue lemma to cases where the oscillating factor appears as the argument of a rational function, we can discuss the high-energy limit of certain very simple multiple scattering problems. Subject to possible convergence difficulties, the discussion can be extended to meromorphic functions of $e^{i\lambda x}$. Unfortunately, we have not found a relation of these ideas to the calculus of residues and are unable to extend the argument to holomorphic functions of $e^{i\lambda x}$. This last extension would be particularly useful in more realistic multiple scattering problems, where operators of the form $(1 - V_1G_0V_2)$ must be inverted.[3]

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[1] A good reference for all these matters is M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1965), esp. Chaps. 3 and 6.

[2] E.T. Whittaker and G.N. Watson, *Modern Analysis* (Cambridge U.P., Cambridge, 1952), expecially Secs. 3.64 and 9.41.

[3] The results of that analysis will be published elsewhere. They are contained in P. B. Kantor, "Scattering From A Composite System; High Energy Limit of the Closure Approximation," Case Western Reserve University, Cleveland, Ohio, Preprint, 1970.

[4] To see this, simply apply the inequality $V_{fg} \leq V_g \sup|f| + V_f \sup|g|$ to each subinterval in which u is monotonic, and use induction on n.

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Diagrammatic Perturbation Expansion for Ensembles of Random Matrices

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A method for obtaining a perturbation expansion of various eigenvalue distributions corresponding to a certain class of perturbed ensembles of random matrices is given. The terms in the expansion can be written down immediately as diagrams analogous to those used in other kinds of perturbation theory. Further, part of the expansion can be summed explicitly, and the result of the summation read off the diagrams. In addition, a new perturbing ensemble is introduced. It has the advantage that the number of matrix elements which are perturbed simultaneously and the size of the perturbation can be varied independently. The expansion given is an expansion in the number of perturbed matrix elements rather than the usual expansion in the size of the elements. Finally, the conditions for convergence of the expansion are discussed.

1. INTRODUCTION

There has been recent interest in the problem of how a small perturbation to the Hamiltonian of a complex system effects the statistical properties of the energy levels.[1–6] The primary purpose of such work is to ascertain the possibility of determining whether or not a particular quantity is an exact invariant or only an approximate invariant by measurement of these statistical properties. Of particular concern is the question of how a small time-reversal invariant term in a Hamiltonian would manifest itself in the statistical properties of the energy spectrum.[2,4,6]

The ensemble which has received the most attention is a simple generalization of the Gaussian ensemble. That is, the unperturbed ensemble is assumed to be Gaussian, say of width $\alpha^{-\frac{1}{2}}$, and the perturbing ensemble is also assumed to be Gaussian, say of width $\gamma^{-\frac{1}{2}}$. Then it is assumed that the relative strength of the perturbation is given by $(\alpha/\gamma)^{\frac{1}{2}}$. One is, of course, interested in the limit where α/γ is small.

In studying this ensemble one encounters mathematical difficulties which seem to be inherent in it. In particular, if one considers the case where the unperturbed distribution is orthogonal (time-reversal invariant) and the perturbing distribution is unitary (not time-reversal invariant), it appears that the results will be purely unitary or orthogonal in the lim $N \to +\infty$ (N is the dimensionality of the matrices) unless $\gamma \to +\infty$ with N in exactly the right way.[2,6] This results from the fact that the number of nonvanishing off-diagonal matrix elements is of order N^2. The convergence of a perturbation expansion in general depends not only on the size of the off-diagonal elements, but also on the number of such elements, since each succeeding term in the expansion involves another summation.

However, the physical interpretation of letting γ