Is Hilbert Space Too Large?

Paul B. Kantor
Tantalus Inc., Cleveland, Ohio 44118, and Case-Western Reserve University, Cleveland, Ohio 44106
(Received 31 December 1985)

Two aspects of the irreversible measurement transformation are explored. The first is the range of allowed values for the joint uncertainty of two operators having no common eigenvector. The second is the constraint on states and operations following from the requirement that the uncertainty resolved by completing a measurement shall be always finite.

PACS numbers: 03.65.Bz

We report an inquiry based on the concept that "measurement" brings about an irreversible change in the quantum-mechanical density matrix describing a physical state.[1] In particular, the entropy measure of a state ρ relative to a set of possibilities $1, 2, \ldots,$ increases under the measurement transformation

$$\rho \rightarrow \sum_i \pi_i \operatorname{Tr}(\rho\pi_i), \tag{1}$$

where π_i is the projection operator corresponding to the ith possibility. In this framework we consider (1) the relation between the entropy measures corresponding to two operators having no eigenstate in common (the Heisenberg uncertainty problem), and (2) possible restrictions on both states and operators following from the requirement that all physical systems and all allowed descriptions of them have finite uncertainty.

In what follows the entropy measure is defined by

$$S(p_1, p_2, \ldots) = -\sum_i p_i \ln p_i; \quad \sum p_i = 1, \tag{2}$$

and the uncertainty of an operator A defined by

$$A = \sum_i a_i \pi_i, \tag{3}$$

in a state ρ is given by

$$U(A,\rho) = S(\operatorname{Tr}(\pi_1\rho), \operatorname{Tr}(\pi_2\rho), \ldots). \tag{4}$$

Note that the uncertainty has (almost) nothing to do with the "physically interesting" numbers $\{a_i\}$ which are the spectrum of A. This presents some puzzling problems mentioned in Sect. 3 below.

1. The Heisenberg uncertainty relation.—There is an attractive relation between the product of the variances of two operators A,B and the expectation value of their commutator, based on the fact that

$$\langle\psi|(A+itB)^\dagger(A+itB)|\psi\rangle \geq 0;$$

specifically, with the variance defined as

$$\langle\psi|\operatorname{var}(A)|\psi\rangle = \langle\psi|(A - \langle\psi|A|\psi\rangle)^2|\psi\rangle, \tag{5}$$

the inequality is

$$\langle\psi|\operatorname{var}(A)\operatorname{var}(B)|\psi\rangle \geq \tfrac{1}{4}|\langle\psi|[A,B]|\psi\rangle|^2. \tag{6}$$

In addition, for household distributions, such as the Gaussian, the variance is an excellent measure of imprecision. Thus Eq. (6) captures, for those distributions, both the spirit and the substance of noncommutativity. For states that are not so localized in a single region of the spectrum, Eq. (6) is less satisfying. Can the uncertainty measure tell us more?

Deutsch[2] has shown that if two operators A and B have no common eigenvector then the joint uncertainty $(U(A,\rho), U(B,\rho))$ cannot assume the value $(0,0)$. In fact, he has shown that the sum $U(A,\rho) + U(B,\rho)$ is bounded below. This leads to the suggestion that the fact that A and B are not simultaneously measurable may be characterized by a measure which we donote as $D(A,B)$, given by

$$D(A,B) \approx \min_\rho [U(A,\rho) + U(B,\rho)]. \tag{7}$$

In an ingenious discussion of the x-p uncertainty relation, Blankenbecler and Partovi[3,4] have implicitly used a somewhat different measure. They construct bins B_i in the variables. Let $P(x,a) = \operatorname{Prob}(x \in B_a)$ and $P(p,b) = \operatorname{Prob}(p \in B_b)$. Blankenbecler and Partovi (BP) implicitly use the measure $M_{\rm BP}(x,p)$ defined by

$$M_{\rm BP}(x,p) = 1 - P(x,a)P(p,b) \tag{8}$$

as a measure of the impossibility of simultaneous measurement. BP fix the bins, and assume that the appropriate choice for the state ρ is the one that has maximum entropy for given values of $P(x,a)$ and $P(p,b)$. Thus the quest for values of $M_{\rm BP}(x,p)$ close to 0 is reflected into a study of the bins B_a and B_b. In this way BP have restored the "physical content" of the impossibility of simultaneous measurement, while retaining the language of projectors and probabilities.

Our research suggests that neither $D(A,B)$ nor $M_{\rm BP}(x,p)$ has a compelling claim to represent that underlying situation. Such a claim would exist if one discovered either function to be the boundary of some allowed region in the plane $(U(A,\rho), U(B,\rho))$ or $(P(x,a), P(p,b))$, respectively.

Specifically, consider the simplest case in which two operators can fail to commute. Choose a representation in which the eigenvectors of A form the basis.

Without loss of generality, the density matrix may be represented as

$$\rho = \begin{Vmatrix} \cos^2\theta & t\sin\theta\cos\theta \\ t^*\sin\theta\cos\theta & \sin^2\theta \end{Vmatrix}, \quad |t| \le 1. \tag{9}$$

The constraint $|t| \le 1$ ensures that the eigenvalues of ρ are positive. Further, without loss of generality, the eigenvectors of B may be taken as

$$b_1 = \begin{Vmatrix} \cos\phi \\ \sin\phi \end{Vmatrix}, \quad b_2 = \begin{Vmatrix} -\sin\phi \\ \cos\phi \end{Vmatrix}. \tag{10}$$

The discussion is simplified by the use of the function

$$g(x) = -x\ln x - (1-x)\ln(1-x). \tag{11}$$

In terms of this function, the uncertainties of the two operators, in the given state, are

$$U(A,\rho) = g(\cos^2\theta), \tag{12a}$$

$$\cos^2\theta_B = \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + 2t\cos\theta\sin\theta\cos\phi\sin\phi, \tag{12b}$$

$$U(B,\rho) = g(\cos^2\theta_B). \tag{12c}$$

The uncertainty $U(B,\rho)$ takes its extreme values for $|t| = 1$, or for t such that the argument of g in Eq. (12b) is 0.5, that is, for t satisfying

$$t = \frac{0.5 - \cos^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi}{2\cos\theta\cos\phi\sin\theta\sin\phi}. \tag{13}$$

The corresponding limits on the allowed points in the joint uncertainty plane are shown in Fig. 1(a) for $\phi = 0.05\pi/2$ and Fig. 1(b) for $\phi = 0.5\pi/2$. The allowed region is not always convex but, as Deutsch has shown, it is bounded by a line on which the sum of the uncertainties is constant.

The shape and size of the allowed region are determined by the transformation matrix

$$M(A,B)_{ij} = \langle a_i | b_j \rangle. \tag{14}$$

For this case, the form of the allowed region is very simple when the variables θ, ϕ are used, as shown in Fig. 2.

We have not established a general result, but conjecture that with a suitable parametrization of the density matrix, this simplicity will be maintained, with ϕ representing the smallest phase of an eigenvalue of the transformation matrix of Eq. (14). Determination of the allowed region may be written as a constrained nonlinear optimization problem, but this formulation obscures the fact that the geometry of the allowed values of $(U(A,\rho), U(B,\rho))$ depends solely on the geometric properties of the transformation matrix.

2. Is Hilbert space too large?—The requirement that $U(A,\rho)$ be finite is a restriction of physically allowed states, when the operator A is given. This restriction follows from two lines of argument. (A) It is unreasonable to suppose that any measurement "yields" an infinite amount of information. (B) More formally, Kantor[5] has argued that information is labeled by the mass-energy physical observable. An infinite amount of information would represent an "immovable object."

Against (A) one may argue that we have lived without this constraint, and quantum mechanics works

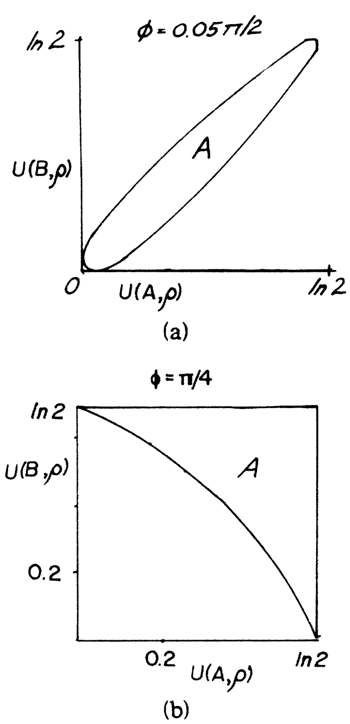

FIG. 1. (a) The allowed values of the uncertainty for $\phi = 0.05\pi/2$. (b) The allowed values of the uncertainty for $\phi = 0.5\pi/2$.

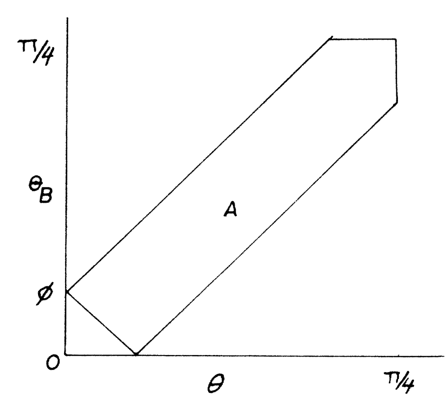

FIG. 2. The allowed region in terms of the variables θ, θ_B.

well. However, a review of the states used in successful calculations reveals that they drop off rapidly at large "distances" and are likely to have finite information when the definition is suitably extended to continuous spectra. Against (B) one may argue that Kantor's formal structure, information mechanics, insists upon conservation of information, does not accommodate the collapse of the wave function, and cannot be applied to the present discussion.

Thus, requiring that $U(A,\rho)$ be finite may not be a physical necessity. Let us explore it nonetheless.

If $U(A,\rho)$ must be finite then Hilbert space is "too large." Specifically, there are vectors in Hilbert space (which we shall realize as square-summable sequences of complex numbers)

$$\psi = (\alpha_1, \alpha_2, \ldots), \tag{15}$$

$$p_i = |\alpha_i|^2, \tag{16}$$

for which

$$\sum p_i = 1, \tag{17}$$

but $S(\psi)$ defined by

$$S(\psi) = -\sum p_i \ln p_i \tag{18}$$

is unbounded.

The existence of such vectors follows from the fact[6] that the sum $Q(\epsilon)$ defined by

$$Q(\epsilon) = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1+E}} \tag{19}$$

converges for $\epsilon > 0$ and diverges for $\epsilon \leq 0$. When the p_n are defined by

$$p_n = 1/n(\ln n)^{1+\epsilon}, \quad n \geq 2, \tag{20}$$

the uncertainty is given by

$$S(\psi) = -\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\epsilon}} - (1+\epsilon) \sum_{n=2}^{\infty} \frac{\ln(\ln n)}{n(\ln n)^{1+\epsilon}}. \tag{21}$$

This expression diverges for $0 < \epsilon \leq 1$.

This establishes that requiring the uncertainty to be finite is a nontrivial constraint. All systems with finitely many degrees of freedom meet this constraint. For systems with infinitely many degrees of freedom we suppose that there corresponds to each observable a sequence of normalized, mutually orthogonal eigenvectors:

$$A \leftrightarrow \{|a_i\rangle\}_{i=1}^{\infty}, \tag{22}$$

$$B \leftrightarrow \{|b_i\rangle\}_{i=1}^{\infty}. \tag{23}$$

(The extension of the entropy measure to the case of the continuous spectrum introduces infinities in its own right, and will not be considered here.) The spectra of the operators A,B will not be considered here, except to suppose that they are nondegenerate.

We will say that a state ψ is of finite uncertainty with respect to A, or

$$(\psi f A) \text{ if } S(|\langle a_1|\psi\rangle|^2, |\langle a_2|\psi\rangle|^2, \ldots) \tag{24}$$

is finite.

We will say that an operator A is of finite uncertainty with respect to B, or

$$(AfB) \text{ if } \forall_i (|a_i\rangle fB). \tag{25}$$

That is, AfB if all of the eigenstates of A are of finite uncertainty with respect to B.

It is clear that AfA. We have been unable to resolve, by either proof or counterexample, the questions of (1) symmetry: Does AfB imply BfA? and (2) transitivity: Do AfB and BfC imply AfC? If the answers to both are affirmative, then there exist equivalence classes of operators with mutually finite uncertainty, and physics as we know it can proceed under the requirement of finite uncertainty. One could seek the subspace of Hilbert space consisting of states having finite uncertainty with respect to all the operators in a given equivalence class. This subspace will be closed under scalar multiplication and finite linear combinations but will not be Cauchy complete.

We propose that, if one is to take the uncertainty measure $U(A,p)$ as a physical property, the essential next steps are to resolve questions (1) and (2) for some case of physical interest, and find the space of states of finite uncertainty. It may be that equivalence classes exist, even if the properties (1) and (2) do not hold universally.

3. Where is the physics?—Physics concerns itself with, broadly, the movement of matter in space-time. In quantum mechanics the familiar concepts of classical physics [position, energy-momentum (but not time)] appear as eigenvalues of operators. Our picture of reality is formulated in terms of them. We say that the particle described by ψ_1 is near that described by ψ_2 if both are fairly concentrated, and the ordinary Euclidean distance between $\langle\psi_1|\mathbf{x}|\psi_2\rangle$ and $\langle\psi_2|\mathbf{x}|\psi_2\rangle$ is not large. The discussion of the measurement transformation loses this fact. If x is replaced by any single-valued function of itself, the effect on ψ of the measurement is unchanged. The "uncertainty" of the pair (x,p) is the same as that for any single-valued functions, $(f(x),g(p))$, however bizarre. This is difficult to accept.

On the other hand, acceptance of the variance as a measure of precision would be naive. Consider the problem of locating a particle which is somewhere among a large collection of boxes. If the boxes are fixed in space, we might find that the particle is almost always in box 15 ($p_{15} \sim 1$) and otherwise nearby (say, in boxes 14 and 16). The uncertainty is low, and so is the variance. If the boxes are shuffled, the uncertainty remains the same, but the variance loses all sense. Perhaps the shuffling could be ruled out by a deeper understanding of its impact on the conjugate translation operator, which now becomes an ugly permutation. The entropic uncertainty relations for a translation operator and its conjugate variable have been discussed by Bialynicki-Birula, who has obtained bounds of the linear form described by Deutsch, for both unbounded and periodic translations.[7]

Even at the naive physical level, are we to say that a particle is "more localized" if it is known to be "somewhere in Rhode Island" or "in the trunk of some car"? The latter is a smaller region, but is not easily characterized. At the simplest level, we may ask the following: If a "particle" is confined to the line interval [0,1], is there a way of determining whether it is in the disjoint bin $[0.0,0.25]\cup[0.5,0.75]$ that does not require more energy than determining whether it is in the bin [0.0,0.5]?

All the aspects of this inquiry are in an unsatisfying state. That is not the fault of several collegues who have given time to listen and comment, especially F. W. Kantor and R. Blankenbecler. This research is part of a broader investigation of implications of the maximum entropy principle, supported in part by the National Science Foundation, Division of Information Science, under Grant No. IST 8318630 to Tantalus, Inc.

[1]John von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, N.J., 1955).

[2]David Deutsch, Phys. Rev. Lett. **50**, 631 (1983).

[3]Richard Blankenbecler and M. Hossein Partovi, Phys. Rev. Lett. **54**, 373 (1985).

[4]M. Hossein Partovi, Phys. Rev. Lett. **50**, 1883 (1983).

[5]Frederick W. Kantor, *Information Mechanics* (Wiley-Interscience, New York, 1977).

[6]J. Pierpont, *Functions of a Complex Variable* (Dover, New York, 1959), p. 47.

[7]Iwo Bialynicki-Birula, in *Quantum Probability and Applications II,* edited by L. Accardi and W. von Waldenfels (Springer-Verlag, New York, 1985).